THE USE OF UNIQUE STATISTICAL WEIGHTS FOR ESTIMATING VARIANCES WITH THE BALANCED HALF-SAMPLE TECHNIQUE

Stanley Lemeshow, University of Massachusetts

This paper considers the use of the balanced half-sample method for estimating the variance of an estimated mean when each individual from a selected sample has been assigned a unique statistical weight. Such individual weighting is an attempt to bring the contribution of sample individuals falling into specified demographic classes into closer alignment with known population figures. Previous work by Kish and Frankel (1968, 1970) and Simmons and Baird (1968) concerned the use of such weights in complex sample surveys such as the Health Examination Survey of the National Center for Health Statistics.

1. Estimating the variance of $\hat{\mu}$ with unique statistical weights

For simplicity a population composed of L strata is considered. Let N be the total number of individuals in this population and let N_i be the number of individuals in the i^{th} stratum, $i = 1, \cdots, L$. Suppose each individual in this population can be classified into one of D demographic groups and let M_j be the number of individuals in the j^{th} demographic group, $j = 1, \cdots, D$. Let N_{ij} be the number of individuals in the i^{th} stratum belonging to the j^{th} demographic group. The population totals, N_i, M_j, $i = 1, \cdots, D$ are know.

Population

Strata \ Demo Group	1	2	$\cdots$	D	
1	N_{11}	N_{12}	$\cdots$	N_{1D}	N_1
2	N_{21}	N_{22}	$\cdots$	N_{2D}	N_2
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
L	N_{L1}	N_{L2}	$\cdots$	N_{LD}	N_L
	M_1	M_2	$\cdots$	M_D	N

Clearly, $\sum_{i=1}^{L} N_i = \sum_{j=1}^{D} M_j = \sum_{i=1}^{L} \sum_{j=1}^{D} N_{ij} = N$,

$\sum_{j=1}^{D} N_{ij} = N_i$, $i = 1, \cdots, L$, $\sum_{i=1}^{L} N_{ij} = M_j$,

$j = 1, \cdots, D$.

Now suppose a random sample of n_i individuals is drawn with replacement from the i^{th} stratum, $i=1, \cdots, L$. Let $n = \sum_{i=1}^{L} n_i$ be the total number of individuals selected from the population. Further, suppose that upon selection each individual is placed, at random, into one of two groups and the demographic class to which the individual belongs is noted. Let n_{ijk} be the number of individuals in the j^{th} demographic class within the i^{th} stratum who were placed at random into the k^{th} group, $k = 1, 2$.

Let $m_{j1} = \sum_{i=1}^{L} n_{ij1}$, $m_{j2} = \sum_{i=1}^{L} n_{ij2}$, $m_j = \sum_{i=1}^{L} n_{ij}$,

$j = 1, 2, \cdots, D$, $n_{ij} = \sum_{k=1}^{2} n_{ijk}$, $i = 1, \cdots, L$,

$j = 1, \cdots, D$.

Sample

Group 1

Strata \ Demo	1	2	$\cdots$	D	
1	n_{111}	n_{121}	$\cdots$	n_{1D1}	$\frac{n_1}{2}$
2	n_{211}	n_{221}	$\cdots$	n_{2D1}	$\frac{n_2}{2}$
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
L	n_{L11}	n_{L21}	$\cdots$	n_{LD1}	$\frac{n_L}{2}$
	m_{11}	m_{21}	$\cdots$	m_{D1}	$\frac{n}{2}$

Group 2

Strata \ Demo	1	2	$\cdots$	D	
1	n_{112}	n_{122}	$\cdots$	n_{1D2}	$\frac{n_1}{2}$
2	n_{212}	n_{222}	$\cdots$	n_{2D2}	$\frac{n_2}{2}$
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
L	n_{L12}	n_{L22}	$\cdots$	n_{LD2}	$\frac{n_L}{2}$
	m_{12}	m_{22}	$\cdots$	m_{D2}	$\frac{n}{2}$

Total Sample

Strata \ Demo	1	2	$\cdots$	D	
1	n_{11}	n_{12}	$\cdots$	n_{1D}	n_1
2	n_{21}	n_{22}	$\cdots$	n_{2D}	n_2
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
L	n_{L1}	n_{L2}	$\cdots$	n_{LD}	n_L
	m_1	m_2	$\cdots$	m_D	n

Note that $\sum_{i=1}^{L} n_i = \sum_{j=1}^{D} m_j = \sum_{i=1}^{L}\sum_{j=1}^{D} n_{ij} = n$, $\sum_{j=1}^{D} n_{ij} = n_i$, $i = 1, \ldots, L$. We also note that since the division into groups is done at random, m_{jk} is usually not equal to $m_j/2$, $k = 1, 2$.

Let $w_{ijk\ell}$ be a statistical weight assigned to the ℓ^{th} individual in the k^{th} group of the j^{th} demographic class in the i^{th} stratum. The $w_{ijk\ell}$ are selected so that following properties hold:

$$\text{(i)} \quad \sum_{i=1}^{L}\sum_{j=1}^{D}\sum_{k=1}^{2}\sum_{\ell=1}^{n_{ijk}} w_{ijk\ell} = N \tag{1}$$

$$\text{(ii)} \quad \sum_{j=1}^{D}\sum_{k=1}^{2}\sum_{\ell=1}^{n_{ijk}} w_{ijk\ell} = N_i \ , \ i=1, \ldots, L \tag{2}$$

$$\text{(iii)} \quad \sum_{i=1}^{L}\sum_{k=1}^{2}\sum_{\ell=1}^{n_{ijk}} w_{ijk\ell} = M_j \ , \ j=1, \ldots, D \tag{3}$$

Let us assume that the demographic breakdown in each stratum is the same. That is,

$$N_{ij} = \frac{N_i \, M_j}{N} \ .$$

Then, by taking

$$W_{ijk\ell} = \frac{N_i \, M_j}{N(n_{ij})} \ , \tag{4}$$

it is easily verified that (1), (2), and (3) hold. Hence, constructing statistical weights in this manner satisfies the desired properties.

Let $x_{ijk\ell}$ be a measurement taken on the ℓ^{th} individual in the k^{th} random group of the j^{th} demographic class in the i^{th} stratum and let μ_{ij} be the mean of the individuals in the j^{th} demographic class in the i^{th} stratum. Let $\mu = \frac{1}{N}\sum_{i=1}^{L}\sum_{j=1}^{D} N_{ij}\,\mu_{ij}$ be the mean in the entire population. Then, μ may be estimated using the statistical weights as follows:

$$\hat{\mu} = \frac{\sum_{i=1}^{L}\sum_{j=1}^{D}\sum_{k=1}^{2}\sum_{\ell=1}^{n_{ijk}} w_{ijk\ell} \ x_{ijk\ell}}{\sum_{i=1}^{L}\sum_{j=1}^{D}\sum_{k=1}^{2}\sum_{\ell=1}^{n_{ijk}} w_{ijk\ell}} \ . \tag{5}$$

We observe that since the n_{ij} are random variables, $\hat{\mu}$ is not a linear estimate but is, instead, a combined ratio estimate.

For this estimate, the denominator of $\hat{\mu}$ equals N. That is

$$\hat{\mu} = \frac{1}{N}\sum_{i=1}^{L}\sum_{j=1}^{D}\sum_{k=1}^{2}\sum_{\ell=1}^{n_{ijk}} w_{ijk\ell} \ x_{ijk\ell}$$

Even here, $\hat{\mu}$ is not a linear estimate since the n_{ij} occur in each of the weights. In cases to be discussed later, where the estimation of the mean of a smaller subgroup in the population is considered, the denominator may not be constant.

Now, we note that

$$E\,(\hat{\mu} | n_{ij} \neq 0) = \mu .$$

Clearly, unbiased estimates of the mean of any collection of the demographic classes of individuals ("domain of interest") can be obtained, subject to the condition that the total of the n_{ijk} in this collection $\neq 0$, by making use of the indicator random variable $\gamma_{ijk\ell}$ as follows:

Let

$$\gamma_{ijk\ell} = \begin{cases} 1 & \text{if individual belongs to the domain of interest} \\ 0 & \text{if individual belongs to some other domain.} \end{cases}$$

Then, the combined ratio estimate

$$\hat{\mu}_D = \frac{\sum_{i=1}^{L}\sum_{j=1}^{D}\sum_{k=1}^{2}\sum_{\ell=1}^{n_{ijk}} \gamma_{ijk\ell} \ w_{ijk\ell} \ x_{ijk\ell}}{\sum_{i=1}^{L}\sum_{j=1}^{D}\sum_{k=1}^{2}\sum_{\ell=1}^{n_{ijk}} \gamma_{ijk\ell} \ w_{ijk\ell}} \tag{6}$$

can be shown to be an unbiased estimate of the mean (to the extent that the $n_{ij} \neq 0$) of the domain of interest.

Once statistical weights of this nature have been established and the corresponding estimate defined, the problem of estimating the variance remains.

Two ways of adjusting individual weights for the purpose of estimating the variance of $\hat{\mu}$ are considered. The first, which we will refer to as the "single-set method", uses the entire sample, as described above, to construct the individual weights. These weights are then used in all calculations. That is, the same individual weights are used for estimates based on only part of the sample individuals (e.g., a half-sample) as are used for estimates based on the entire sample. Clearly, since the weights were designed to bring the entire sample into closer alignment with the population, the resulting estimates based on a fraction of the sample will not necessarily reflect the population parameter. A second method, which we will refer to as the "multiple-set method", considers the subgroup of the sample from which a population estimate is to be computed and assigns weights to the individuals in the subgroup. There is then a closer alignment between specified demographic classes in the subgroup and the population.

a. The Single-Set Method

The single-set method always assigns the individuals the statistical weight

$$w_{ijk\ell} = \frac{N_i M_j}{N(n_{ij})} \quad .$$

Hence, for the balanced half-sample method, letting δ_{ij} be an element from the appropriate Plackett-Burman matrix, we compute m half-sample estimates $\hat{\mu}_{(p)}$, $p = 1, \cdots, m$, where

$$\hat{\mu}_{(p)} = \frac{\sum_{i=1}^{L} \{\delta_{pi} \sum_{j=1}^{D} \sum_{\ell=1}^{n_{ij1}} w_{ij1\ell}\, x_{ij1\ell} + (1-\delta_{pi}) \sum_{j=1}^{D} \sum_{\ell=1}^{n_{ij2}} w_{ij2\ell}\, x_{ij2\ell}\}}{\sum_{i=1}^{L} \{\delta_{pi} \sum_{j=1}^{D} \sum_{\ell=1}^{n_{ij1}} w_{ij1\ell} + (1-\delta_{pi}) \sum_{j=1}^{D} \sum_{\ell=1}^{n_{ij2}} w_{ij2\ell}\}} \tag{7}$$

and

$$\delta_{pi} = \begin{cases} +1 & \text{if the first group is to be used from the } i^{th} \text{ stratum} \\ 0 & \text{if the second group is to be used from the } i^{th} \text{ stratum.} \end{cases}$$

Once the $\hat{\mu}_{(p)}$ are calculated, the variance of $\hat{\mu}$ may be estimated by using

$$\hat{V}_B(\hat{\mu}) = \frac{1}{m} \sum_{i=1}^{m} (\hat{\mu}_{(i)} - \hat{\mu})^2 \tag{8}$$

where $\hat{\mu}$ is defined as in (5).

b. The Multiple Set Method

Consider a second set of individual statistical weights

$$V_{ijk\ell} = \frac{N_i M_j}{N(n_{ijk})} \quad .$$

Then, for the balanced half-sample method we compute m half sample estimates $\hat{\mu}_{(p)}$, $p = 1, \cdots, m$, where

$$\hat{\mu}_{(p)} = \frac{\sum_{i=1}^{L} \{\delta_{pi} \sum_{j=1}^{D} \sum_{\ell=1}^{n_{ij1}} v_{ij1\ell}\, x_{ij1\ell} + (1-\delta_{pi}) \sum_{j=1}^{D} \sum_{\ell=1}^{n_{ij2}} v_{ij2\ell}\, x_{ij2\ell}\}}{\sum_{i=1}^{L} \{\delta_{pi} \sum_{j=1}^{D} \sum_{\ell=1}^{n_{ij1}} v_{ij1\ell} + (1-\delta_{pi}) \sum_{j=1}^{D} \sum_{\ell=1}^{n_{ij2}} v_{ij2\ell}\}} \tag{9}$$

We note that $E(\hat{\mu}_{(p)} | n_{ijk} \neq 0) = \mu$. Hence, $\hat{\mu}_{(p)}$, $p = 1, \cdots, m$, is an unbiased estimate of μ provided all $n_{ijk} > 0$.

The balanced half-sample estimates of the variance of $\hat{\mu}$ are defined as in (8) with the $\hat{\mu}_{(i)}$, $i = 1, \cdots, m$, defined as in (9).

It is often of interest to estimate the variance of $\hat{\mu}$ for certain domains of interest in the population. That is, we might want to estimate the variance of the estimated mean, $\hat{\mu}_D$, of one of the demographic classes into which the population is divided. As before, an indicator variable, $\gamma_{ijk\ell}$, which takes on the value 1 if the individual belongs to the demographic class for which an estimate is desired and 0 otherwise, is used. Then, the p^{th} half-sample estimate of $\hat{\mu}_D$ is

$$\hat{\mu}_{D(p)} = \frac{\sum_{i=1}^{L} \{\delta_{pi} \sum_{j=1}^{D} \sum_{\ell=1}^{n_{ij1}} \gamma_{ij1\ell}\, w^*_{ij1\ell}\, x_{ij1\ell} + (1-\delta_{pi}) \sum_{j=1}^{D} \sum_{\ell=1}^{n_{ij2}} \gamma_{ij2\ell}\, w^*_{ij2\ell}\, x_{ij2\ell}\}}{\sum_{i=1}^{L} \{\delta_{pi} \sum_{j=1}^{D} \sum_{\ell=1}^{n_{ij1}} \delta_{ij1\ell}\, w^*_{ij1\ell} + (1-\delta_{pi}) \sum_{j=1}^{D} \sum_{\ell=1}^{n_{ij2}} \gamma_{ij2\ell}\, w^*_{ij2\ell}\}} ,$$

where

$$w^*_{ijk\ell} = \begin{cases} w_{ijk\ell} & \text{if the single-set method is used} \\ v_{ijk\ell} & \text{if the multiple-set method is used.} \end{cases}$$

Then, the balanced half-sample estimate of the variance of $\hat{\mu}_D$ is

$$\hat{V}_B(\hat{\mu}_D) = \frac{1}{m} \sum_{i=1}^{m} (\hat{\mu}_{D(i)} - \hat{\mu}_D)^2,$$ where $\hat{\mu}_D$ is defined as in (6).

2. The Sampling Experiment

Sampling experiments were performed using both the single-set or multiple-set weights in conjunction with the balanced half-sample method to estimate the variance of $\hat{\mu}$.

In the sampling experiments, n_i observations were randomly selected from the i^{th} stratum, $i = 1, \cdots, L$. On the j^{th} draw from the i^{th} stratum we observe the random pair (x_{ij}, u_{ij}) where

$$X_{ij} \sim N(\mu^{(i)}, \sigma^{(i)}_{xx})$$

and

$$U_{ij} \sim U(0,1) \quad .$$

The population is categorized into D = 2 demographic classes, and the probability of falling into either of the two is specified. That is, let

$$P_1 = \frac{M_1}{N}$$

be the proportion of the population belonging to the first demographic class. This proportion is a specified constant for all strata.

Then, if

$$u_{ij} \begin{cases} < p_1, & \text{the individual is a member of the first demographic class} \\ > p_1, & \text{the individual is a member of the second demographic class.} \end{cases}$$

We label the first demographic class "domain 1" and the second "domain 2". Hence, for each of the n observations drawn from this population we have a value for x as well as the demographic class to which the observation belongs.

The means and variances of x are specified for each of the L strata. Also specific are N, $p_1=M_1/N$, N_i and n_i, $i = 1, \ldots, L$. We will be interested in obtaining estimates of the variance of the estimated mean of the entire population, of doman 1 alone and of domain 2 alone.

The following "situations" were considered:

Situation (i-1) $L = 3$, $\mu_x^{(1)}=5$, $\mu_x^{(2)}=10$, $\mu_x^{(3)}=15$, $\sigma_{xx}^{(i)}=1$, $i=1,2,3$, $p_1=.8$, $N_1=1000$, $N_2=2000$, $N_3=7000$, $N=10000$, $n_i=100$, $i=1,2,3$.

Situation (i-2): Situation (i-1) with $n_i=10$, $i=1,2,3$.

Situation (i-3): Situation (i-1) with $n_i=20$, $i=1,2,3$.

Situation (i-4): Situation (i-1) with $n_i=50$, $i=1,2,3$.

Situation (i-5): Situation (i-1) with $p_1=.9$

Situation (i-6): " " " $p_1=.7$

Situation (i-7): " " " $p_1=.6$

Situation (i-8): " " " $p_1=.5$

Situation (i-9): " " " $N_1=3333$, $N_2=3333$, $N_3=3334$.

Situation (i-10): Situation (i-1) with $N_1=7000$, $N_2=2000$, $N_3=1000$

Situation (i-11): Situation (i-1) with $\mu^{(i)}=10$, $i=1,2,3$

Situation (i-12): Situation (i-1) with $\mu^{(1)}=5$, $\mu^{(2)}=6$, $\mu^{(3)}=7$

Situation (ii) $L=15$, $\mu_x^{(1)}=5$, $\mu_x^{(i)}=\mu_x^{(1)} + 5(i-1)$, $i=2,\ldots,15$, $\sigma_{xx}^{(i)}=1$, $i=1,\ldots,15$, $p_1=.8$, $N_1=300$, $N_i=N_1 + 100(i-1)$, $i=2,\ldots,15$, $N=15000$, $n_i=n=100$, $i=1,\ldots,L$.

For situations (i-1) - (i-11), 1000 repetitions of the experiment were taken and for situation (ii), 200 repetitions were taken.

The variance of $\hat{\mu}$ was estimated with $\hat{V}_B(\hat{\mu})$.

Table 1 presents the estimated absolute relative bias[1] and variance[2], of the estimated variance of $\hat{\mu}$ which resulted from each of the sampling experiments performed for the thirteen situations under consideration. Results are presented using both the single-set and multiple-set methods for the more prevalent domain 1 individuals, the less prevalent domain 2 individuals, and for all sample individuals. Considered in this table, as described earlier, is an L strata situation where we vary the number of observations selected in each stratum, the probability of selecting an individual from domain 1, the population sizes of the strata, the means of the strata, or the number of strata.

Situations (i-1) through (i-10) correspond to having $\mu^{(1)} = 5$, $\mu^{(2)} = 10$, $\mu^{(3)} = 15$. That is, these strata are rather spread out - particularly since $\sigma_{xx}^{(i)} = 1$, $i = 1,2,3$. As is seen in Table 1, when we estimate the variance of μ for specific domains of interest using the single-set method with the balanced half-sample technique, the resulting variance estimates show extremely high bias and variability when compared to the same estimates using the multiple-set method. In these situations, use of $v_{ijk\ell}$ is advantageous since they depend on the n_{ijk} instead of the n_{ij}. The latter were used to construct the $w_{ijk\ell}$. When calculations are performed on the entire sample, the number of individuals in each stratum is equally divided among the two established groups and there no longer appears to be an advantage to using the multiple-set method.

We note that when groups are established within each stratum, n_{ijk}, the number of individuals belonging to a specific demographic class within a group, is a random variable. That is, n_{ijk} is not necessarily $n_{ij}/2$. In fact, when a small number of observations are selected from each stratum it is possible, as is indicated by the "*", that $n_{ijk} = 0$ in some cases.

In situations (i-11) and (i-12) where the means in each stratum are either equal (all $\mu^{(i)}= 5$) or close, ($\mu^{(i)} = 5,6,7$, $\sigma_{xx}^{(i)} = 1$), there appears to be no advantage to using the multiple-set method.

Calculations with situation (ii) indicate that, with 15 strata with widely differing means, it is certainly advisable to use the multiple-set method. For domain 1, the absolute relative bias was reduced from 60.157 using the single-set method to .054 using the multiple-set method. For domain 2, the absolute relative bias was reduced from 221.663 using the single-set method to .045 using the multiple-set method. In fact, even for calculations involving all sample individuals, for these 15 highly spread strata, an absolute relative-bias of 3.038 using the single-set method was reduced to .112 using the multiple-set method. The variances of the variance estimates using the single-set method were much higher than the corresponding variances using the multiple-set method.

4. Conclusions

Our conclusion , therefore, is that the preferred procedure is to weight the individuals in each half-sample so that the half-sample estimates are representative of the population. We believe that an explanation for the similarity of the variance estimates produced by the two weighting methods found by Kish and Frankel (1968, 1970) and Simmons and Baird (1968) might be that:

(a) Calculations were not performed on domains of interest-particularly in which the n_{ijk} in each group were likely to be small, and

(b) The means of the strata are not likely to be very different for that particular set of data.

We believe that when variances are estimated for domains of interest - particularly when the strata means are not likely to be similar - it is advisable to reweight each half-sample. We note that, as was seen in formula (9), this does not involve the establishment of m+1 sets of weights (one for the total sample and one for each of the m half-samples). Instead, only two weights must be computed for each sample individual. The amount of extra work necessary in using this method, especially with the speed of modern computing facilities, is minimal.

By using the multiple-set weighting method, adequate variance estimates, in most of the situations considered in this sampling experiment, were obtained. Use of the single-set method led to an extremely high degree of bias and variability in certain of the situations considered. However, it should be recognized that these results only apply to the case of random sampling with replacement from L strata. For this reason, further research is necessary to determine the appropriateness of the single-set or multiple-set methods for more complex survey designs such as those used in the Health Examination Survey.

Table 1: Estimated values based on sampling experiments are presented for the absolute relative bias, $|\{\hat{E}[\hat{V}_B(\hat{\mu}_D)] - \tilde{V}(\hat{\mu}_D)\}/\tilde{V}(\hat{\mu}_D)|$, and variance, $\hat{V}(\hat{V}_B(\hat{\mu}_D))$ for domain 1, domain 2 and all sample individuals. Estimates are made using the single-set and multiple-set method of individual weighting in each of the thirteen situations described in the text.

ABSOLUTE RELATIVE BIAS

	DOMAIN 1		DOMAIN 2		TOTAL SAMPLE	
Situation	Single -Set	Multiple -Set	Single -Set	Multiple -Set	Single -Set	Multiple -Set
(i-1)	.949	.115	5.438	.209	.094	.124
(i-2)	1.355	.009	*	*	.116	.212
(i-3)	1.488	.136	*	*	.274	.397
(i-4)	1.069	.059	6.213	1.172	.041	.033
(i-5)	.474	.111	9.507	3.268	.063	.077
(i-6)	1.402	.113	4.192	.132	.092	.126
(i-7)	2.118	.110	3.619	.102	.079	.116
(i-8)	2.734	.047	2.988	.053	.087	.108
(i-9)	3.305	.080	14.772	.166	.099	.064
(i-10)	1.089	.069	5.624	.251	.025	.058
(i-11)	.117	.115	.125	.209	.149	.124
(i-12)	.068	.115	.303	.209	.147	.124
(ii)	60.157	.054	221.663	.045	3.038	.112

VARIANCE

	DOMAIN 1		DOMAIN 2		TOTAL SAMPLE	
Situation	Single -Set	Multiple -Set	Single -Set	Multiple -Set	Single -Set	Multiple -Set
(i-1)	.00026	.00007	.04763	.00199	.00005	.00004
(i-2)	.06322	.01098	*	*	.01098	.00791
(i-3)	.01032	.00243	*	*	.00246	.00257
(i-4)	.00112	.00032	.88485	.58082	.00022	.00024
(i-5)	.00012	.00006	2.49803	2.13784	.00005	.00005
(i-6)	.00053	.00009	.01190	.00076	.00005	.00004
(i-7)	.00118	.00013	.00472	.00041	.00005	.00004
(i-8)	.00204	.00023	.00218	.00021	.00005	.00005
(i-9)	.00032	.00001	.07618	.00028	.00001	†
(i-10)	.00025	.00006	.04748	†	.00004	.00004
(i-11)	.00007	.00007	.00144	†	.00004	.00004
(i-12)	.00008	.00007	.00190	†	.00004	.00004
(ii)	.00104	†	.43840	†	.00002	†

* some $n_{ijk} = 0$

† value < .00001

ENDNOTES

[1]The absolute relative bias of the estimated variance of $\hat{\mu}$ can be estimated from a sampling experiment as follows. For each repetition of the sampling experiment $\hat{\mu}$ is computed as if $\hat{V}_B(\hat{\mu})$, the estimated variance of $\hat{\mu}$ using the balanced half-sample technique. Then, let

(i) $E(\hat{\mu})$ = expected value of $\hat{\mu}$. This is estimated as $\tilde{E}(\hat{\mu}) = \frac{1}{r} \sum_{i=1}^{r} \hat{\mu}_i$, where r is the number of repetitions of the experiment and $\hat{\mu}_i$ is the estimate of μ for the i^{th} repetition.

(ii) $V(\hat{\mu})$ = variance of $\hat{\mu}$. This is estimated as $\tilde{V}(\hat{\mu}) = \frac{1}{r-1} \sum_{i=1}^{r} (\hat{\mu}_i - \tilde{E}(\hat{\mu}))^2$.

With r large, this value is regarded as the "target value" of the variance estimator $\hat{V}_B(\hat{\mu})$.

(iii) $E(\hat{V}_B(\hat{\mu}))$ = expected value of the half-sample estimation technique. This is estimated by

$$\tilde{E}[\hat{V}_B(\hat{\mu})] = \frac{1}{r} \sum_{i=1}^{r} \hat{V}_{B_i}(\hat{\mu}), \text{ where } \hat{V}_{B_i}(\hat{\mu})$$

is the half-sample estimate of the variance of $\hat{\mu}$ produced during the i^{th} repetition of the sampling experiment.

The absolute relative bias is defined as

$$\left| \frac{E[\hat{V}_B(\hat{\mu})] - V(\hat{\mu})}{V(\hat{\mu})} \right| .$$

This is estimated by

$$\left| \frac{\tilde{E}(\hat{V}_B(\hat{\mu}) - \tilde{V}(\hat{\mu})}{\tilde{V}(\hat{\mu})} \right| .$$

This relative measure places the bias into the context of the target value. It is the estimated bias as a proportion of the estimated target value.

[2]The variance of the balanced half-sample estimates of the variance of $\hat{\mu}$, $V[\hat{V}_B(\hat{\mu})]$ is estimated by

$$\tilde{V}[\hat{V}_B(\hat{\mu})] = \frac{1}{r-1} \sum_{i=1}^{r} \{\hat{V}_{B_i}(\hat{\mu}) - \tilde{E}[\hat{V}_B(\hat{\mu})]\}^2 .$$

ACKNOWLEDGMENTS

This research is based, in part, on the author's Ph.D. dissertation which was undertaken at UCLA. In this regard, computing assistance was obtained from the Health Sciences Computing Faculty (UCLA) under NIH Special Resources Grant RR-3. Other support was obtained at the University of North Carolina (Chapel Hill) through the U.S. Bureau of the Census under Joint Statistical Agreement JSA 76-93.

REFERENCES

Kish, Leslie and Frankel, Martin R. (1968). Balanced repeated replication for analytical statistics. Proceedings of the Social Statistics Section of the American Statistical Association: 2-10.

______________ (1970). Balanced repeated replication for standard errors. Journal of the American Statistical Association, Vol. 65, No. 331: 1071-1094.

Simmons, Walt R. and Baird, James T., Jr. (1968). Pseudo-replication in the NCHS Health Examination Survey. Proceedings of the Social Statistics Section of the American Statistical Association: 19-30.